Differential Equations

Question Paper

Level	Pre U
Subject	Maths
Exam Board	Cambridge International Examinations
Topic	Differential Equations
Booklet	Question Paper

Time Allowed: **102 minutes**

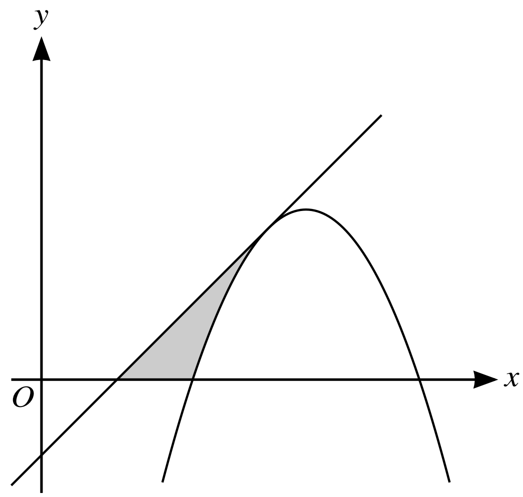
Score: **/85**

Percentage: **/100**

Grade Boundaries:

1 Solve the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = \sec y$ given that $y = \frac{\pi}{6}$ when $x = 4$ giving your answer in the form $y = \mathrm{f}(x)$. [6]

2 The diagram shows the curve with equation $y = 7x - 10 - x^2$ and the tangent to the curve at the point where $x = 3$.



(i) Show that the curve crosses the x-axis at $x = 2$. [1]

(ii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence find the equation of the tangent to the curve at $x = 3$.

Show that the tangent crosses the x-axis at $x = 1$. [5]

(iii) Evaluate $\int_2^3 (7x - 10 - x^2)\,\mathrm{d}x$ and hence find the exact area of the shaded region bounded by the curve, the tangent and the x-axis. [7]

3 A differential equation is given by $2\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - y)$.

(i) Express $\frac{2}{y(1-y)}$ in partial fractions. [3]

(ii) Hence show by integration that $\frac{y^2}{(1-y)^2} = A\mathrm{e}^x$. [5]

(iii) Given that $x = 0$ when $y = 2$, find the value of A and express y in terms of x. [3]

4 A new lake is stocked with f sh. Let P_t be the population of f sh in the lake after t years. Two models using recurrence relations are proposed for P_t, with $P_0 = 550$.

$$\text{Model 1}: P_t = 2P_{t-1}e^{-0.001P_{t-1}}$$
$$\text{Model 2}: P_t = \tfrac{1}{2}P_{t-1}\left(7 - \tfrac{1}{160}P_{t-1}\right)$$

(i) Evaluate the population predicted by each model when $t = 3$. [4]

(ii) Identify, with evidence, which one of the models predicts a stable population in the long term. [2]

(iii) Describe the long term behaviour of the population for the other model. [1]

5 The table below gives the population of breeding pairs of red kites in Yorkshire from 2001 to 2008.

Year	2001	2002	2003	2004	2005	2006	2007	2008
Number of breeding pairs	8	10	16	24	33	40	47	69

Source: www.gigrin.co.uk

The following model for the population has been proposed:

$$N = a \times b^t,$$

where N is the number of breeding pairs t years after the year 2000, and a and b are constants.

(i) Show that the model can be transformed to a linear relationship between $\log_{10} N$ and t. [2]

(ii) On graph paper, plot $\log_{10} N$ against t and draw by eye a line of best fit. Use your line to estimate the values of a and b in the equation for N in terms of t. [6]

(iii) What values of N does the model give for the years 2008 and 2020? [2]

(iv) In which year will the number of breeding pairs first exceed 500 according to the model? [3]

(v) Comment on the suitability of the model to predict the population of breeding pairs of red kites in Yorkshire. [1]

6 A tank with vertical sides and rectangular cross-section is initially full of water. The water is leaking out of a hole in the base of the tank at a rate which is proportional to the square root of the depth of the water. V m^3 is the volume of water in the tank at time t hours.

(i) Show that $\frac{dV}{dt} = a\sqrt{V}$, where a is a constant. [2]

(ii) Given that the tank is half full after one hour, show that $V = V_0\left(\left(\frac{1}{\sqrt{2}} - 1\right)t + 1\right)^2$, where V_0 m^3 is the initial volume of water in the tank. [7]

(iii) Hence show that the tank will be empty after approximately 3 hours and 25 minutes. [2]

7 Solve the differential equation $\frac{dy}{dx} = -y^2x^3$, where $y = 2$ when $x = 1$, expressing your solution in the form $y = f(x)$. [6]

8 Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x(1 + x^2)}$$

giving your answer in the form $y = f(x)$. [10]

9 Diane is given an injection that combines two drugs, Antiflu and Coldcure. At time t hours after the injection, the concentration of Antiflu in Diane's bloodstream is $3e^{-0.02t}$ units and the concentration of Coldcure is $5e^{-0.07t}$ units. Each drug becomes ineffective when its concentration falls below 1 unit.

(i) Show that Coldcure becomes ineffective before Antiflu. [3]

(ii) Sketch, on the same diagram, the graphs of concentration against time for each drug. [2]

(iii) 20 hours after the first injection, Diane is given a second injection. Determine the concentration of Coldcure 10 hours later. [2]