ZNOTES.ORG

UPDATED TO 2020-21 SYLLABUS

CAIE AS LEVEL
FURTHER MATHS (9231)

SUMMARIZED NOTES ON THE FURTHER PURE 1 SYLLABUS

1. Roots of Polynomial Equations

1.1. Coefficients of Polynomials

- Quadratic Equations $(\mathbf{ax^2+bx+c=0})$
  - $\sum\alpha=\alpha+\beta=-\frac{b}{a}$
  - $\sum\alpha\beta=\alpha\beta=\frac{c}{a}$
  - $S_n=\alpha^n+\beta^n$
- Cubic Equations $(\mathbf{ax^3+bx^2+cx+d=0})$
  - $\sum\alpha=\alpha+\beta+\gamma=-\frac{b}{a}$
  - $\sum\alpha\beta=\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$
  - $\sum\alpha\beta\gamma=\alpha\beta\gamma=-\frac{d}{a}$
  - $S_n=\alpha^n+\beta^n+\gamma^n$
- Quartic Equations $(\mathbf{ax^4+bx^3+cx^2+dx+e=0})$
  - $\sum\alpha=\alpha+\beta+\gamma+\delta=-\frac{b}{a}$
  - $\sum\alpha\beta=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$
  - $\sum\alpha\beta\gamma=\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}$
  - $\sum\alpha\beta\gamma\delta=\alpha\beta\gamma\delta=\frac{e}{a}$
  - $S_n=\alpha^n+\beta^n+\gamma^n+\delta^n$
- Recurrence Notation
  - $\sum\alpha$ is also known as S_1
  - $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\gamma$ is also known as S_2
  - $\sum\frac{1}{\alpha}$ is also known as S_{-1}. It's always equal to the negative of the coefficient of the linear term divided by the coefficient of the constant term

1.2. Algebraic Combinations

- Finding an equation through algebraic manipulation

Ex 1.3 Question 2b:
$\mathbf{x^3+3x^2-2x+5=0}$ has roots $\alpha,\ \beta,\ \gamma$
Find equation with roots $(\alpha-\mathbf{1}),\ (\beta-\mathbf{1}),\ (\gamma-\mathbf{1})$

Solution:

Using coefficients:

1. $\alpha+\beta+\gamma=-\mathbf{3}$
2. $\alpha\beta+\beta\gamma+\alpha\gamma=-\mathbf{2}$
3. $\alpha\beta\gamma=-\mathbf{5}$

Equation to find:

1. $(\alpha-\mathbf{1})+(\beta-\mathbf{1})+(\gamma-\mathbf{1})=$

$$=\alpha+\beta+\gamma-\mathbf{3}$$

$$\therefore -\mathbf{3}-\mathbf{3}=\mathbf{6}$$

1. $(\alpha-\mathbf{1})(\beta-\mathbf{1})+(\alpha-\mathbf{1})(\gamma-\mathbf{1})+(\beta-\mathbf{1})(\gamma-\mathbf{1})$

$$=\alpha\beta-\alpha-\beta+\mathbf{1}+\alpha\gamma-\alpha-\gamma+\mathbf{1}+\beta\gamma-\beta-\gamma+\mathbf{1}$$

$$=(\alpha\beta+\alpha\gamma-\beta\gamma)-\mathbf{2}(\alpha+\beta+\gamma)+\mathbf{3}$$

$$\therefore\ -\mathbf{2}-\mathbf{2}(-\mathbf{3})+\mathbf{3}=\mathbf{7}$$

1. $(\alpha-\mathbf{1})(\beta-\mathbf{1})(\gamma-\mathbf{1})$

$$=(\alpha\gamma-\alpha-\beta+\mathbf{1})(\gamma-\mathbf{1})$$

$$=\alpha\beta\gamma-\alpha\beta-\alpha\gamma+\alpha-\beta\gamma+\beta+\gamma-\mathbf{1}$$

$$=\alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-\mathbf{1}$$

$$\therefore -\mathbf{5}-(-\mathbf{2})+(-\mathbf{3})+\mathbf{1}=-\mathbf{7}$$

Thus, equation is: $\mathbf{x^3-6x^2+7x+7=0}$

1.3. Substitution

- For Finding sums of roots to a specific degree of power

{SP20-P01} Question 4:
The Cubic Equation $z^3-z^2-z-5=0$ has roots α, β and γ

1. **Show that $\alpha^3+\beta^3+\gamma^3=19$**
2. **Find the value of $\alpha^4+\beta^4+\gamma^4$**

Solution:
We can use Recurrence Notation since:

$$S_3=\alpha^3+\beta^3+\gamma^3$$

Find S_1**and **S_2. **From the polynomial:*

$$S_1=1$$

Using $S_2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$

$$S_2=(1)^2-2(-1)=3$$

From our polynomial, we know that:

$$S_3-S_2-S_1-15=0$$

Substitute in values and find S_3

$$S_3-(3)-(1)-15=0$$

$$\to S_3=15+3+1$$

$$\to S_3=19\quad(\text{Ans for a})$$

In order to find S_4, we will need to use a substitution, we will be letting $y=z^2\to z=\sqrt{y}$
Rearrange the terms:

$$z^3-z=z^2+5$$

Square both sides:

$$z^6-2z^4+z^2=z^4+10z^2+25$$

Rearrange the terms back and substitute in $z = \sqrt{y}$:

$$z^6 - 3z^4 - 9z^2 - 25 = 0$$

$$\rightarrow y^3 - 3y^2 - 9y - 25 = 0$$

S_2 in the new equation is S_4 in the old equation

$$\mathbf{S_2} = (\mathbf{3})^{\mathbf{2}} - \mathbf{2}\,(-\mathbf{9}) = \mathbf{27} \quad (\text{Ans for b})$$

2. Rational Functions

2.1. Partial Fractions

- To split an improper fraction into partials, use Polynomial Division and make sure the degree of the numerator is higher than the denominator.
- If the degree of the numerator is less than the denominator, use partial fraction method and equate coefficient.

2.2. Vertical Asymptote

- Making denominator 0 resulting in ∞
- Example:

$$y = \frac{1}{(x+1)(x-3)}$$

Thus, vertical asymptotes at: $x = -1 \text{ and } 3$

2.3. Horizontal Asymptote

- By dividing the top and bottom of a fraction by x, we can see what value y tends to when x becomes very large
- Example:

$$y = \frac{3x-2}{x+1}$$

Divide numerator and denominator by x

$$y = \frac{\frac{3x-2}{x}}{\frac{x+1}{x}} = \frac{3 - \frac{2}{x}}{1 + \frac{1}{x}}$$

When x is very large, $y = \frac{3}{1} = 3$
Thus, horizontal asymptote at: $y = 3$

2.4. Oblique Asymptotes

- Occurs only with improper fraction
- Example:

$$y = 2x - 1 + \frac{2}{x-1} - \frac{3}{x+2}$$

When x becomes very large, $y \approx 2x - 1$
Thus, oblique asymptote at: $y = 2x - 1$

2.5. Sign Tables

- Used to visualize graph as it shows in which quadrant the graph lies
- Enter values of x which result in different parts of the fraction equaling zero
- Leave columns between each value of x and place signs to indicate whether value +ve or –ve in each cell
- Example:

$$y = \frac{3x^2 + 3x + 6}{(x+3)(x-2)}$$

x		- 3		2	
3x^{2} + 3x + 6	+	+	+	+	+
x + 3	-	0	+	+	+
x - 2	-	-	-	0	+
y	+	infinity	-	infinity	+

2.6. Range of Function

- Discriminant:
  - $b^2 - 4ac = 0$: Tangent
  - $b^2 - 4ac < 0$: Lines do not meet (are not in range)
  - $b^2 - 4ac > 0$: Lines do meet (are in range)
- We can use the discriminant to show the Range of the function. Most of the time we use $b^2 - 4ac < 0$ to show what values of y that does not exist to then show what values of y exists.

2.7. Curve Sketching

- When you sketch the curve, include the following:
  - y-intercept
  - x-intercepts
  - Stationary points (maxima, minima, inflections)
  - Vertical asymptote(s)
  - Horizontal or oblique asymptote(s)

{SP20-P01} Question 7:
The Curve C has equation $y = \frac{2x^2-3x-2}{x^2-2x+1}$
a) State the equations of the asymptotes of C
b) Show that $y \leq \frac{25}{12}$ at all points on C
c) Find the coordinates of any stationary points of C
d) Sketch $\mathbf{C}$, stating the coordinates of any intersections of $\mathbf{C}$ with the coordinate axes and the asymptotes

Solution:

In order to find all the asymptotes, we will have to split the improper fraction as much as possible:
Apply polynomial division to our function:

$$y = \frac{2x^2 - 3x - 2}{x^2 - 2x + 1} = 2 + \frac{x+2}{(x-1)^2}$$

As $\mathrm{x} \rightarrow \infty, y = 2$ *which is our horizontal asymptote*

Our vertical asymptote will be when the denominator =0, which means:

$$(x-1)^2 = 0$$

$$x = 1$$

Therefore, the equations of asymptotes are:

$$y = 2, \quad x = 1 \quad (\text{Ans for a})$$

To show the range of C, we will be using the discriminant $b^2 - 4ac < 0$ to show the values where y do not exist. Rearrange terms:

$$y = \frac{2x^2 - 3x - 2}{x^2 - 2x + 1}$$

$$\to x^2y - 2xy + y = 2x^2 - 3x - 2$$

$$\to (y-2)\,x^2 + (3-2y)\,x + (2+y) = 0$$

Applying discriminant:

$$(3-2y)^2 - 4\,(y-2)\,(2+y) < 0$$

$$\to 25 \;-\; 12y \;< 0$$

$$y > \frac{25}{12}$$

*We found the Range of values for y that DO NOT EXIST, this implies the range of values where *y exists is at :

$$y \le \frac{25}{12} \quad (\text{Ans for b})$$

Of course, we will have to differentiate the function in order to find the stationary points, differentiating it we get:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{7-x}{(x-1)^3}$$

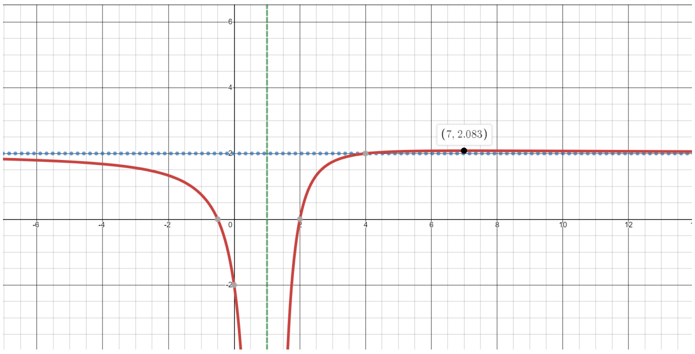
Equate $\frac{\mathrm{dy}}{\mathrm{dx}} = 0$:

$$\frac{7-x}{(x-1)^3} = 0$$

$$\to x = 7$$

Putting the x-value to the function to give its y-value, we get $y = \frac{25}{12}$. Which is just the highest point in our graph.

$$\left(7,\ \frac{25}{12}\right) \quad (\text{Ans for c})$$

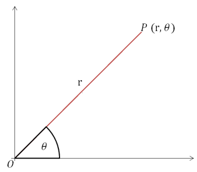


(Ans for d)

3. Polar Coordinates

3.1. Definitions

- A point, P, has coordinates $(r, 0)$ where:
  - r is the distance from the pole, O
  - θ is angle measure from base half line to radius OP



- Important points:
  - $(r,\ \theta)$is an ordered pair – must always be in that order
  - Angle always measured positive anticlockwise, principal value which is $-\pi < \theta < \pi$
  - Angle measured in radians
  - r can only be positive

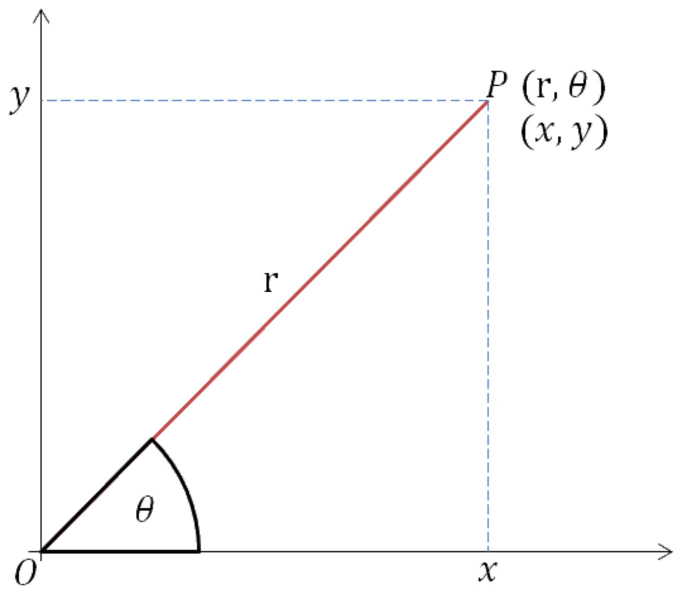
3.2. Converting between Cartesian and Polar

- Basic facts:

$x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$

- Represented on a diagram:



{S04-P01}Question 3:
The curve $\mathbf{C}$ has equation

$$\left(\mathbf{x}^2+\mathbf{y}^2\right)^2=\mathbf{4xy}$$

Show that the polar equation of $\mathbf{C}$ is $\mathbf{r}^2=\mathbf{2}\sin\mathbf{2}\theta$

Solution:

Using identities, form an equation in terms of $\mathbf{r}$ and θ

$$\left(\mathbf{r}^2\right)^2=\mathbf{4}(\mathbf{r}\cos\theta)(\mathbf{r}\sin\theta)$$

$$\mathbf{r}^4=\mathbf{4r}^2\cos\theta\sin\theta$$

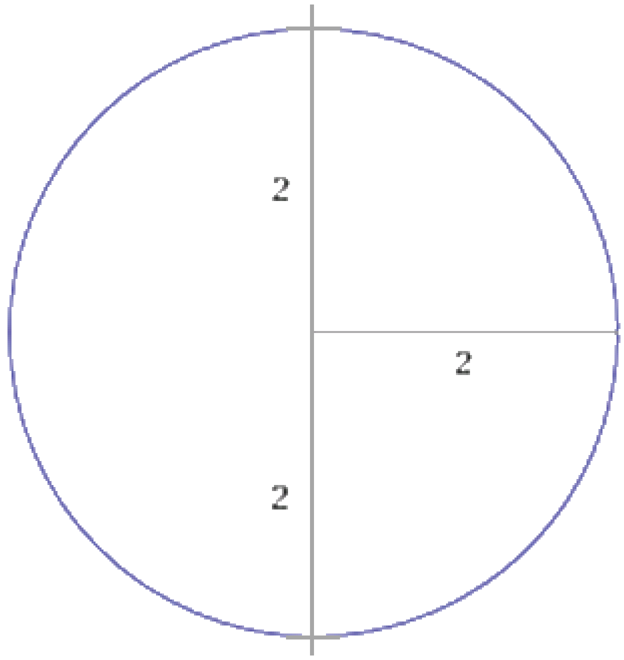
$$\mathbf{r}^2=\mathbf{2}(\mathbf{2}\cos\theta\sin\theta)$$

$$\mathbf{r}^2=\mathbf{2}\sin\mathbf{2}\theta$$

3.3. Sketching Polar Curves

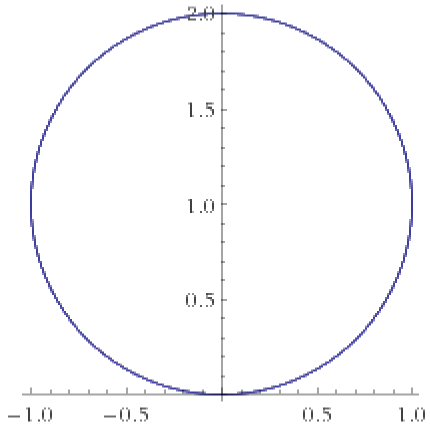
Circles

- $r=a$
  - Radius is a
  - Centre of circle: $(0,\ 0)$



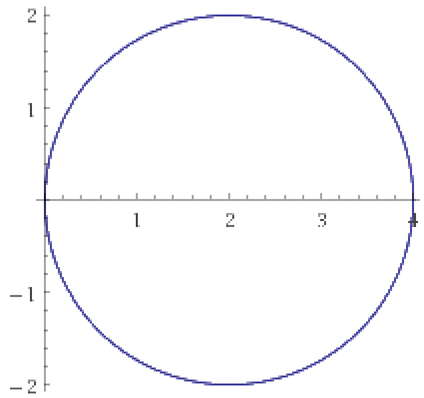
$r=2$

- $r=a\sin\theta$
  - Diameter is a
  - Centre of circle: $\left(\frac{a}{2},\frac{\pi}{2}\right)$



$r=2\sin\theta$

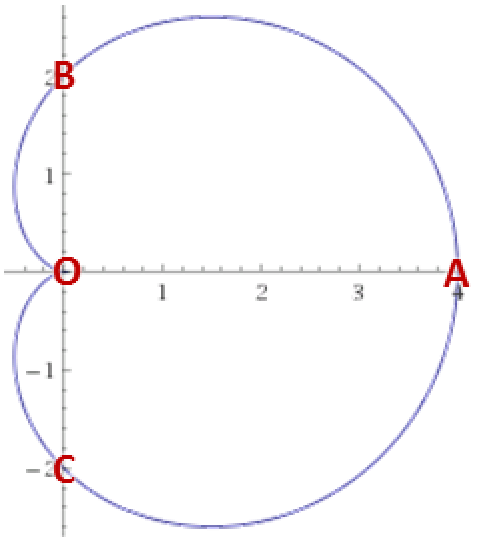
- $r=a\cos\theta$
  - Diameter is a
  - Centre of circle: $\left(\frac{a}{2},\ 0\right)$



$r = 4\cos\theta$

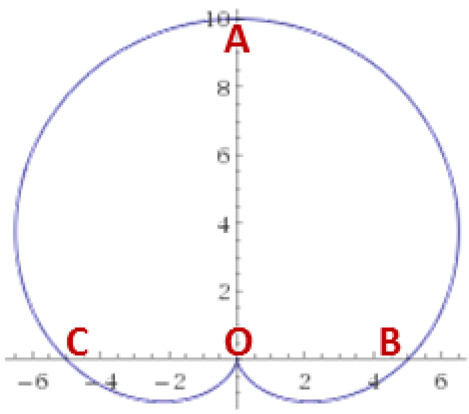
Cardioids

- $r = a + a\cos\theta$
  - $|\mathrm{OA}| = 2a$
  - $|\mathrm{OB}| = |\mathrm{OC}| = a$



$r = 2 + 2\cos\theta$

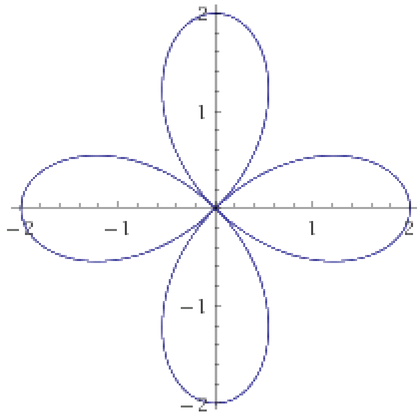
- $r = a + a\sin\theta$
  - $|\mathrm{OA}| = 2a$
  - $|\mathrm{OB}| = |\mathrm{OC}| = a$



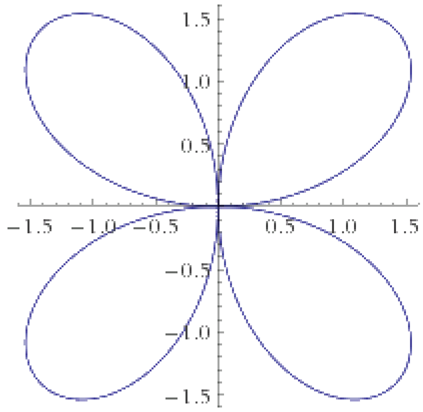
$r = 5 + 5\sin\theta$

Flowers

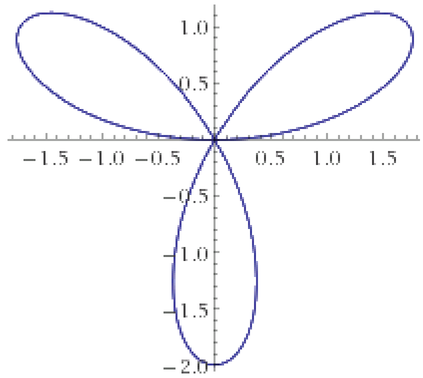
- $r = a\cos \mathrm{b}\theta$ or $a\sin \mathrm{b}\theta$
  - Length of petal $= a$
  - No. of petals:
    - if b odd then b petals
    - if b even then $2b$ petals
  - Cosine flower graphs start from$\theta = 0°$ line
  - Sine flower graphs start from $\theta = 45°$ line



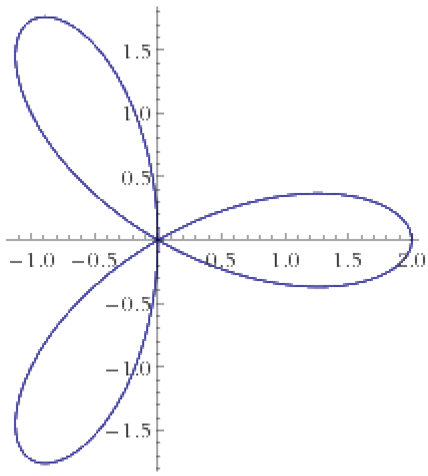
$r = 2\cos 2\theta$



$r = 2\sin 2\theta$



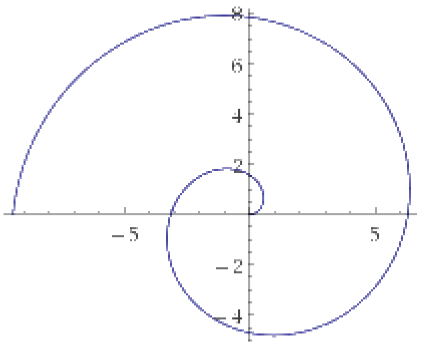
$r = 2\sin 3\theta$



$$r = 2\cos 3\theta$$

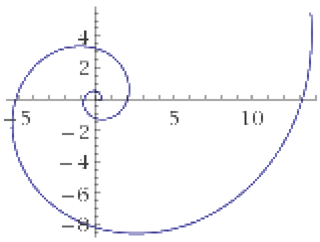
Spirals

- When sketching spirals, first recognize the type they are, locate the centre and find intersections at $n\left(\frac{\pi}{2}\right)$
- $r = a\theta$
    - $a > 1$ looser spiral
    - $a < 1$ tighter spiral
    - Begins at $(0,\ 0)$



$$r = \theta$$

- $r = ae^{b\theta}$
    - First intersection to origin $= a$
    - Begins at $(a, 0)$



3.4. Extremes – Maxima, Minima & Tangents

1. Furthest point from origin
    1. Maximize/Minimize: r
    2. Find: $\frac{dr}{d\theta}$
2. Horizontal Tangent
    1. Maximize/Minimize: $y = r\ sin\theta$
    2. Find: $\frac{dx}{d\theta}$
3. Vertical Tangent
    1. Maximize/Minimize: $x = r\ cos\theta$
    2. Find: $\frac{dy}{d\theta}$

3.5. Calculus in Polar Curves

- Area enclosed by a curve:

$$\int \frac{1}{2} r^2 \mathrm{d}\theta$$

- Length of an arc:

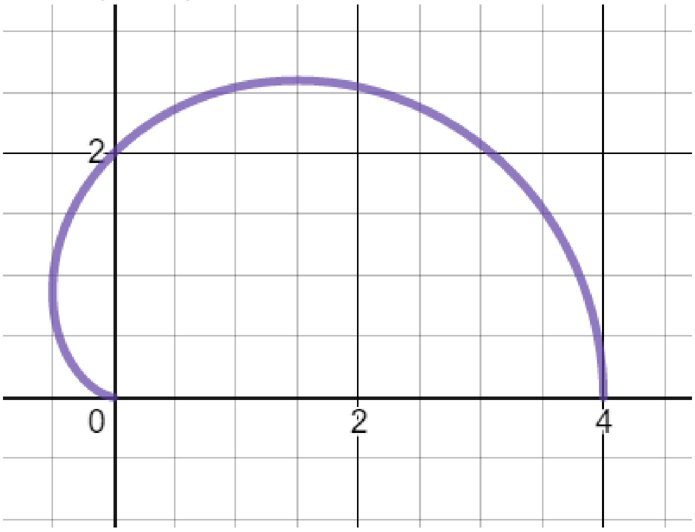
$$\int \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \quad \text{(Not needed for AS level)}$$

{SP20-P01} Question 3
The curve C has a polar equation $r = 2 + 2\cos\theta$, for $0 \leq \theta \leq \pi$
a) Sketch C
b) Find the area of the region enclosed by C and the initial line**
c) Show that the Cartesian equation of C can be expressed as $4\left(x^2 + y^2\right) = \left(x^2 + y^2 - 2x\right)^2$

Solution:

We can refer to Section 3.3 for sketching cardioids to help us draw.

Sketching C, we get:



(Ans for a)

Use the formula to find area of a Polar Curve:

$$\int_0^{\pi} \frac{1}{2}(2 + 2\cos\theta)^2 \ \mathrm{d}\theta$$

$$\rightarrow \int_0^{\pi} \frac{1}{2}\left(4 + 8\cos\theta + 4\cos^2\theta\right) \mathrm{d}\theta$$

$$\rightarrow \int_0^{\pi} \left(2 + 4\cos\theta + 2\cos^2\theta\right) \ \mathrm{d}\theta$$

Use the double angle formula:

$$\rightarrow \int_0^{\pi} 2 + 4\cos\theta + 1 + \cos 2\theta d\theta$$

$$\rightarrow \left[3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$\rightarrow 3\pi + 4\sin\pi + \frac{1}{2}\sin 2\pi = 3\pi \quad \text{(Ans for b)}$$

Using the formula shown in Section 3.2, let $r^2 = x^2 + y^2,\ \ x = r\cos\theta$* and $y = r\sin\theta$.*

$$r = 2 + 2\cos\theta$$

$$\rightarrow \sqrt{x^2+y^2} = 2 + 2 \times \frac{x}{\sqrt{x^2+y^2}}$$

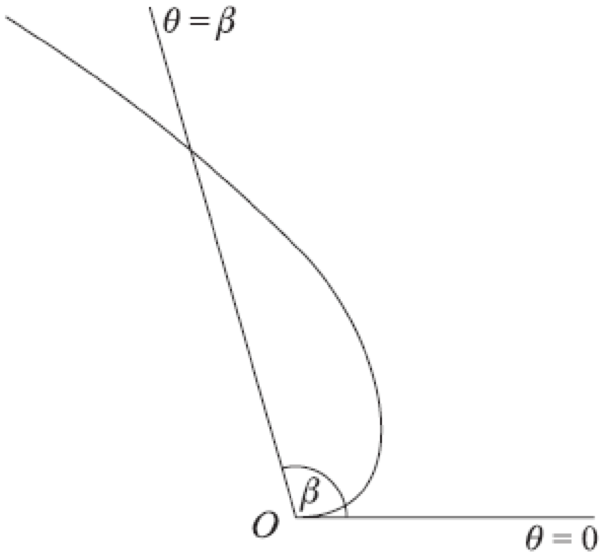
$$\rightarrow x^2+y^2 = 2\sqrt{x^2+y^2} + 2x$$

$$\rightarrow 2\sqrt{x^2+y^2} = x^2+y^2-2x$$

Squaring both sides:

$$4\left(x^2+y^2\right) = \left(x^2+y^2-2x\right)^2 \quad (\text{Ans for c})$$

{S03-P01} Question 1:



The curve C has polar equation

$$r = \theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}}$$

where $0 \le \theta \le \pi$. The area of the finite region bounded by $\mathbf{C}$ and the line $\theta = \beta$ is π. Show that

$$\beta = (\pi \ln 3)^{\frac{1}{2}}$$

Solution:

Form an equation by using the area of sector formula

$$\int \frac{1}{2} r^2 \theta.\, d\theta = A$$

$$\int_0^\beta \frac{1}{2}\left(\theta^{\frac{1}{2}} e^{\frac{\theta^2}{\pi}}\right)^2 \theta = \pi$$

Take ½ to the other side and clean up

$$\int_0^\beta \theta e^{\frac{2\theta^2}{\pi}} = 2\pi$$

Integrate the expression with respect to θ

$$\left[\frac{\pi}{4} e^{\frac{2\theta^2}{\pi}}\right]_0^\beta = 2\pi$$

Take constant to other side and substitute β *and 0*

$$e^{\frac{2\beta^2}{\pi}} - 1 = 8$$

$$e^{\frac{2\beta^2}{\pi}} = 9$$

Take $\ln \mathrm{e}$ *on both sides and simplify*

$$\frac{2\beta^2}{\pi} = \ln 9$$

$$\beta^2 = \pi \ln 3$$

$$\therefore \beta = (\pi \ln 3)^{\frac{1}{2}}$$

4. Mathematical Induction

4.1. Proof by Induction

- Step 1: proving assertion is true for some initial value of variable
- Step 2: the inductive step
- Conclusion: final statement of what you have proved

4.2. Proof of Divisibility

{SP20-P01} Question 2:

It is given that $\phi(n) = 5^n(4n+1) - 1$, for $n = 1, 2, 3...$ Prove, by mathematical induction, that $\phi(n)$ is divisible by 8 for every positive integer n

Solution:

Step 1:

Let $n = 1$

$$\therefore \phi(1) = 5(4+1) - 1 = 8 \times 3$$

Our formula is true for $\mathbf{n = 1}$

Step 2:

Assume formula is true for $n = k$ for an integer k

$$\phi(\mathbf{k}) = \mathbf{5^k(4k+1) - 1 = 8p}$$

Where $\mathbf{p}$ *is just a dummy value*

Let $n = k+1$

$$\phi(\mathbf{k+1}) = \mathbf{5^{k+1}(4(k+1)+1) - 1}$$

$$\rightarrow \phi(\mathbf{k+1}) = \mathbf{5^k(5)(4k+5) - 1}$$

$$\rightarrow \phi(\mathbf{k+1}) = \mathbf{5^k(20k+25) - 1}$$

Calculate $\phi(\mathbf{k+1}) - \phi(\mathbf{k})$

$$\phi(\mathbf{k+1}) - \phi(\mathbf{k}) = \mathbf{5^k(20k+25) - 1 - \left(5^k(4k+1) - 1\right.}$$

$$\phi(\mathbf{k+1}) - \phi(\mathbf{k}) = \mathbf{5^k(20k+25) - 1 - 5^k(4k+1) + 1}$$

$$\phi(\mathbf{k+1}) - \phi(\mathbf{k}) = \mathbf{5^k(20k+25-4k-1)}$$

$$\phi(\mathbf{k+1}) - \phi(\mathbf{k}) = \mathbf{5^k(16k+24)}$$

$$\phi(\mathbf{k+1}) - \phi(\mathbf{k}) = \mathbf{5^k(8)(2k+3)}$$

Add $\phi(\mathbf{k})$to both sides

$$\phi(\mathbf{k+1}) = \mathbf{5^k(8)(2k+3)} + \phi(\mathbf{k})$$

From the equation, because $\phi(\mathbf{k})$ *divisible by 8, therefore* $\phi(\mathbf{k+1})$ *is also divisible by 8.*

Conclusion:

Thus, since $\phi(\mathbf{1})$is true and $\phi(\mathbf{k}) \rightarrow \phi(\mathbf{k+1})$ is true by mathematical induction.

4.3. Proof of Summation

{S03-P01}: Question 2:

Prove by induction that, for all $\mathbf{N} \geq \mathbf{1}$,

$$\sum_{\mathbf{n=1}}^{\mathbf{N}} \frac{\mathbf{n+2}}{\mathbf{n(n+1)2^n}} = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{(N+1)2^N}}$$

Solution:

Step 1:

Let $\mathbf{N} = \mathbf{1}$,

$$\frac{\mathbf{1+2}}{\mathbf{1(1+1)2^1}} = \frac{\mathbf{3}}{\mathbf{4}}$$

Using the formula given:

$$\mathbf{1} - \frac{\mathbf{1}}{\mathbf{(1+1)2^1}} = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{4}} = \frac{\mathbf{3}}{\mathbf{4}}$$

Therefore, true for $\mathbf{N} = \mathbf{1}$

Step 2:

Assume formula true for $\mathbf{N} = \mathbf{k}$

When $\mathbf{N} = \mathbf{k}$,

$$\mathbf{1} - \frac{\mathbf{1}}{\mathbf{(k+1)2^k}}$$

When $\mathbf{N} = \mathbf{k+1}$

$$\mathbf{1} - \frac{\mathbf{1}}{\mathbf{(k+1+1)2^{k+1}}} = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{(k+2)2^{k+1}}}$$

If formula is true then,

$$\sum_{\mathbf{n=1}}^{\mathbf{k+1}} \frac{\mathbf{n+2}}{\mathbf{n(n+1)2^n}} = \sum_{\mathbf{n=1}}^{\mathbf{k}} \frac{\mathbf{n+2}}{\mathbf{n(n+1)2^n}} + (\mathbf{k+1})^{\text{th}} \text{ term}$$

$$= \mathbf{1} - \frac{\mathbf{1}}{\mathbf{(k+1)2^k}} + \frac{\mathbf{k+3}}{\mathbf{(k+1)(k+2)2^{k+1}}}$$

$$= \mathbf{1} - \frac{\mathbf{2(k+2) - k - 3}}{\mathbf{(k+1)(k+2)2^{k+1}}}$$

$$= \mathbf{1} - \frac{\mathbf{2k+4-k-3}}{\mathbf{(k+1)(k+2)2^{k+1}}}$$

$$= \mathbf{1} - \frac{\mathbf{k+1}}{\mathbf{(k+1)(k+2)2^{k+1}}}$$

$$= \mathbf{1} - \frac{\mathbf{1}}{\mathbf{(k+2)2^{k+1}}}$$

Conclusion:

By the Principle of Mathematical Induction, the formula is true for all $\mathbf{N} \geq \mathbf{1}$.

4.4. Proof of Derivatives

Example:

Find the nth derivative of xe^x

Solution:

Step 1: Specialise

$$\frac{d}{\mathrm{dx}}(xe^x) = xe^x + e^x$$

$$\frac{d^2}{dx^2}(xe^x) = xe^x + e^x + e^x$$

$$\frac{d^3}{dx^3}(xe^x) = xe^x + e^x + e^x + e^x$$

Step 2: Generalise

We can see that the pattern that for the nth derivative, there are $\boldsymbol{ne^x}$s.

Step 3: Conjecture

$$\frac{d^n}{dx^n}(xe^x) = xe^x + ne^x$$

Step 4: Proof

Let $n = k$,

$$\frac{d^k}{dx^k}(xe^x) = xe^x + ke^x$$

To find $n = k+1$,*differentiate expression:*

$$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = xe^x + e^x + ke^x$$

$$= xe^x + (k+1)e^x$$

Prove that formula gives same result, $n = k+1$,

$$\frac{d^{k+1}}{dx^{k+1}} = xe^x + (k+1)e^x$$

By the Principle of Mathematical Induction,

$xe^x + ne^x$ is the nth derivative of xe^x for all $n \geq 1$.

5. Summation of Series

5.1. Standard Results of Sums

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

5.2. General Summation Rules

$$\sum \text{kr} = k\sum r$$

$$\sum (r+s) = \sum r + \sum s$$

$$\sum_{a}^{b} r = \sum_{1}^{b} r - \sum_{1}^{a-1} r$$

{S04-P01}: Question 1:
Use the relevant standard results in the List of Formulae to prove that

$$S_N = \sum_{n=1}^{N} (8n^3 - 6n^2) = N(N+1)(2N^2-1)$$

Hence show that

$$\sum_{n=N+1}^{2N} (8n^3 - 6n^2)$$

Can be expressed in the form

$$N\left(aN^3 + bN^2 + cN + d\right)$$

Where the constants $a,\ b,\ c,\ d$ are to be determined.

Solution:

Split up using summation rule

$$\sum_{n=1}^{N} (8n^3 - 6n^2) = 8\sum_{n=1}^{N} n^3 - 6\sum_{n=1}^{N} n^2$$

Using standard results of sums

$$= 8\left(\frac{1}{4}n^2(n+1)^2\right) - 6\left(\frac{1}{6}n(n+1)(2n+1)\right)$$

$$= 2n^2(n+1)^2 - n(n+1)(2n+1)$$

$$= n(n+1)(2n(n+1) - (2n+1))$$

$$= n(n+1)(2n^2-1)$$

For the next part, split the summation into two parts

$$= \sum_{n=1}^{2N} (8n^3 - 6n^2) - \sum_{n=1}^{N+1-1} (8n^3 - 6n^2)$$

Using the rule above, substitute and simplify

$$= 2N(2N+1)(8N^2-1) - N(N+1)(2N^2-1)$$

$$= N\left((4N+2)(8N^2-1) - (N+1)(2N^2-1)\right)$$

Expand and simplify

$$= N\left(30N^3 + 14N^2 - 3N - 1\right)$$

5.3. Method of Differences

- In general, telescoping sums are finite sums in which pairs of consecutive terms cancel each other, leaving only the initial and final terms.
- Let a_n be a sequence of numbers. Then,

$$\sum_{n=1}^{N} (a_n - a_{n-1}) = a_N - a_0$$

5.4. Convergence

- Finite series approaches a limit as more terms are added
- One condition is that terms must get smaller
- Satisfying this condition alone is not always sufficient
- We denote it using the following:

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n) = \ldots$$

{SP20-P01} Question 1:
a) Given that $f(r) = \frac{1}{(r+1)(r+2)}$, show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}$$

b) Hence find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$$

c) Deduce the value of

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$$

Solution:

We can directly calculate $f(r-1) - f(r)$ by simply substituting in the function:

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$\rightarrow \frac{r+2}{r\left(r+1\right)\left(r+2\right)} - \frac{r}{r\left(r+1\right)\left(r+2\right)}$$

$$\rightarrow \frac{2}{r\left(r+1\right)\left(r+2\right)} \quad \text{(Ans for a)}$$

Relating to a), we can see that:

$$\sum_{r=1}^{n} \frac{1}{r\left(r+1\right)\left(r+2\right)} = \frac{1}{2}\sum_{r=1}^{n} \frac{1}{r\left(r+1\right)} - \frac{1}{\left(r+1\right)\left(r+2\right)}$$

We will use the Method of Differences as shown in Method of Differences

$r=1$	$+\frac{1}{2}$	$-$	$\frac{1}{6}$
$r=2$	$+\frac{1}{6}$	$-$	$\frac{1}{12}$
$r=3$	$+\frac{1}{12}$	$-$	$\frac{1}{20}$
$\vdots$	$\vdots$	$-$	$\vdots$
$r=n-2$	$+\frac{1}{(n-2)(n-1)}$	$-$	$\frac{1}{(n-1)(n)}$
$r=n-1$	$+\frac{1}{(n-1)(n)}$	$-$	$\frac{1}{n(n+1)}$
$r=n$	$+\frac{1}{n(n+1)}$	$-$	$\frac{1}{(n+1)(n+1)}$

Therefore:

$$\sum_{r=1}^{n} \frac{1}{r\left(r+1\right)\left(r+2\right)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\left(n+1\right)\left(n+1\right)}\right)$$

(Ans for b)

For c), we can set the limit as n goes to infinity:

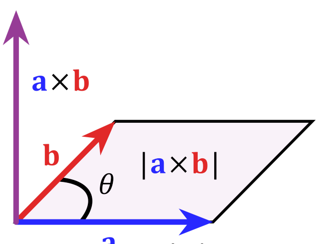
$$\lim_{n\rightarrow\infty} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\left(n+1\right)\left(n+1\right)}\right)$$

$$\rightarrow \frac{1}{2}\left(\frac{1}{2}\right)$$

$$= \frac{1}{4} \quad \text{(Ans for c)}$$

6. Vectors

6.1. Vector Product



* The vector product of two vectors results in the common perpendicular to both vectors

* For two vectors ${a}$ and ${b}$**:**

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \; b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

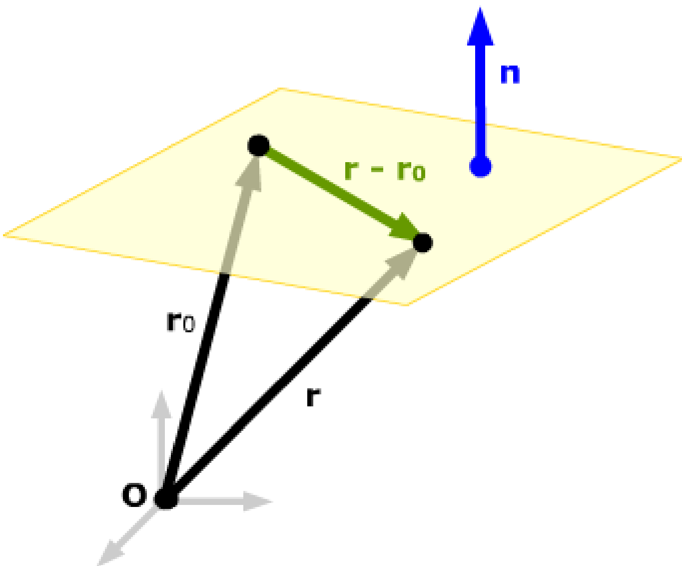
The vector product can be found the determinant of a matrix consisting of the two vectors:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- You can calculate the angle between the two vectors by

$$|a \times b| = |a|\,|b| \sin\theta$$

6.2. Equation of a Plane



- **Parametric form:** a plane is made up of two direction vectors hence can be written as

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1} + \mu\mathbf{d_2}$$

- **Scalar product form:** find the normal vector by finding cross product of the two direction vectors. Find D by substituting a point in $\mathbf{r}$

$$\mathbf{r}.\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = D$$

- **Cartesian form:** coefficients are components of the normal vector

$$n_1x + n_2y + n_1z = D$$

6.3. Finding the Equation of a Plane

- **Given 3 points on a plane:**

$$A = (1, 2, -1)\,,\; B = (2, 1, 0)\,,\; C = (-1, 3, 2)$$

- Find 2 direction vectors e.g. AB and AC (can be any pair) and find the cross product. This is the normal:

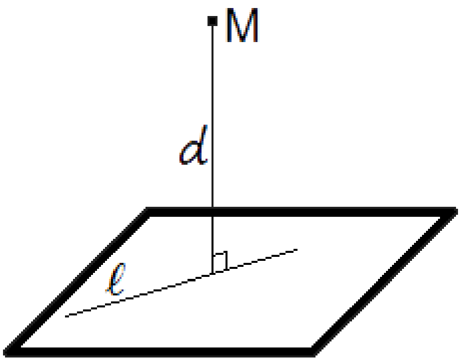
$$\therefore \mathbf{n} = AB \times AC = \begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix}$$

- Substitute point $A$to get D

$$\therefore \mathbf{r}.\begin{pmatrix} -4 \\ -5 \\ -1 \end{pmatrix} = -13$$

- **Given a point and a line on the plane:** Make 2 points on the line by substituting different values for λ. Repeat the 3-point process as above.
- **Given 2 lines on a plane:** Find a point on one line and 2 points on the other line by substituting different values in λ. Repeat the 3-point process as above.

6.4. ⊥ Distance from a Point to a Plane



$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

6.5. Line of Intersection of Two Planes

- The direction of the line of intersection would be normal to both the normal of the plans so

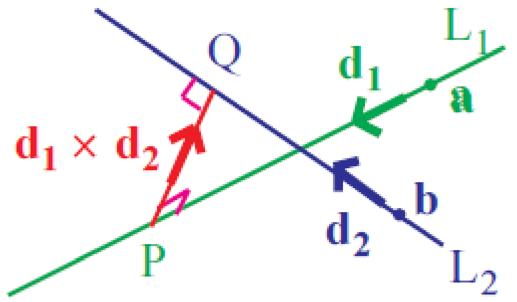
$$\mathbf{d} = \mathbf{n_1} \times \mathbf{n_2}$$

- To find a point on the plane, set one of the variables to a value and solve to find two other points
- *Two ways there is no line of intersection:*
  - Planes may be parallel – if so, normal vectors would be the same (or negative)
  - May be the same plane with different equations

6.6. Intersection of a Line and a Plane

- Find point by substituting line as $\mathbf{r}$ into the scalar product of the plane, find λ and find coordinates
- *Two ways there is no point of intersection:*
  - Line is parallel to the plane – equation with λs won't solve and $\mathbf{d.n} = 0$
  - Line lies in the plane – equation ends with $0 = 0$ and any point on the line will be a solution. Also $\mathbf{d.n} = 0$

6.7. Distance between Two Skew Lines



$L_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1}$ $L_2 : \mathbf{r} = \mathbf{b} + \mu\mathbf{d_2}$

- Observing diagram above, one can follow line L_1 to point P and then moving along the normal of the two lines to point Q. This can be represented by a line as

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1} + t(\mathbf{d_1} \times \mathbf{d_2})$$

- This point can also be reached simply with line L_2. Hence as they both get to the same point; we can equate above line and L_2

$$\mathbf{a} + \lambda\mathbf{d_1} + t\,(\mathbf{d_1} \times \mathbf{d_2}) = \mathbf{b} + \mu\mathbf{d_2}$$

- Form three equations using each coordinate and solve to find λ, t and μ.
- The perpendicular distance required between the two skew lines is $|t\,(\mathbf{d_1} \times \mathbf{d_2})|$
- Shortest distance of two skew lines where both lines are expressed in the form of $\mathbf{r} = \mathbf{a} + \mathbf{b}t$:

$$\frac{(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2})}{|\mathbf{b_1} \times \mathbf{b_2}|}$$

Equation of the Line of the Shortest Distance:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d_1} + t(\mathbf{d_1} \times \mathbf{d_2})$$

- Substitute all the values, simplify, and form the equation with the parameter t

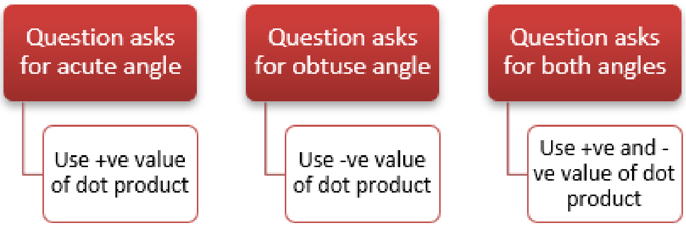
6.8. Angles

$$\frac{\mathrm{a.b}}{|a|\,|b|} = \cos\theta$$

- **Angle between two skew lines:**
  - Dot product between the two direction vectors
  - If vectors in opposite directions, find obtuse angle

- **Angle between line and plane:**
  - Dot product between the line's direction vector and the plane's normal
  - Angle found is with the normal so do $90-\theta$
- **Angle between plane and plane:**
  - Dot product between their normal's
  - If obtuse find equivalent acute

- When using dot product rule to find an angle,



{SP20-P01} Question 6
The position vectors of the points A, B, C, D are

$$2i+4j-3k$$

$$-2i+5j-4k$$

$$i+4j+k$$

$$i+5j+mk$$

respectively, where m is an integer. It is given that the shortest distance between the line through A and B and the line through C and D is 3.
a) **Show that the only possible value of m is 2.**
b) **Find the shortest distance of D from the line through A and C.**
c) **Show that the acute angle between the planes ACD and BCD is $\cos^{-1}\frac{1}{\sqrt{3}}$**

Solution:

$$r_{\text{AB}}=2i+4j-3k+(-4i+j-k)t$$

$$r_{\text{CD}}=i+4j+k+(j+(m-1)k)s$$

Using cross product to find normal vector:

$$n=\begin{vmatrix} i & j & k \\ -4 & 1 & -1 \\ 0 & 1 & m-1 \end{vmatrix}=mi+4(m-1)j-4k$$

Using the formula for the shortest distance of two skew lines:

$$\frac{(a_2-a_1)\cdot(b_1\times b_2)}{|b_1\times b_2|}$$

Equate to 3 ∵ shortest distance is 3

$$\rightarrow\frac{(i-4k)\cdot(mi+4(m-1)j-4k)}{\sqrt{m^2+(4(m-1))^2+(-4)^2}}=3$$

$$\rightarrow\frac{m+16}{\sqrt{m^2+16m^2-32m+16+16}}=3$$

$$\rightarrow\frac{m+16}{\sqrt{17m^2-32m+32}}=3$$

Square both sides and rearrange:

$$m^2+32m+256=9\left(17m^2-32m+32\right)$$

$$\rightarrow-152m^2+320m-32=0$$

Divide both sides by -8

$$19m^2-40m+4=0$$

$$\rightarrow(19m-2)(m-2)=0$$

$$m_1=\frac{2}{19}\qquad m_2=2$$

We take $m=2$because the question says that m is an integer. Therefore:

$$m=2\quad(\text{Ans for a})$$

$$\overrightarrow{\text{OC}}=2i+4j-3k+(-i+4k)t$$

or $(2-t)i+4j+(-3+4t)k$

$$\overrightarrow{\text{OD}}=i+5j+2k$$

Find vector connecting D to line C parametrically:

$$\overrightarrow{\text{DO}}+\overrightarrow{\text{OC}}=(1-t)i-j+(-5+4t)k$$

Use the dot product against the direction vector of AC which is $-i+4k$

$$\overrightarrow{\text{DC}}\cdot(-i+4k)=-(1-t)+4(-5+4t)$$

Equate to 0 since we need the case where $\cos\theta=0$

$$-(1-t)+4(-5+4t)=0$$

$$t=\frac{21}{17}$$

Put the value back into $\overrightarrow{\text{DC}}$and get its distance

$$\left|\overrightarrow{\text{DC}}\right|=\sqrt{\left(1-\frac{21}{17}\right)^2+(-1)^2+\left(-5+4\left(\frac{21}{17}\right)\right)^2}$$

$$\left|\overrightarrow{\text{DC}}\right|=\sqrt{\frac{18}{17}}\approx1.03\quad(\text{Ans for b})$$

Find relevant vectors to find normal of the plane:

$$\overrightarrow{\text{AC}}=-i+4k$$

$$\overrightarrow{\text{AD}}=-i+j+5k$$

$$\overrightarrow{BC} = 3i - j + 5k$$

$$\overrightarrow{BD} = 3i + 6k$$

Find $\overrightarrow{AC} \times \overrightarrow{AD}$ and $\overrightarrow{BC} \times \overrightarrow{BD}$

$\overrightarrow{AC} \times \overrightarrow{AD} = |i \quad j \quad k\ -1 \quad 0 \quad 4\ -1 \quad 1 \quad 5| = -4i + j - k$

$\overrightarrow{BC} \times \overrightarrow{BD} = |i \quad j \quad k\ 3 \quad -1 \quad 5\ 3 \quad 0 \quad 6| = -6i - 3j + 3k$

Use dot product to find angle between the two normals:

$$a \cdot b = |a|\,|b| \cos\theta$$

$$\rightarrow \cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

Substituting values into our formula:

$$\cos\theta = \frac{\left(\overrightarrow{AC} \times \overrightarrow{AD}\right) \cdot \left(\overrightarrow{BC} \times \overrightarrow{BD}\right)}{\left|\overrightarrow{AC} \times \overrightarrow{AD}\right| \left|\overrightarrow{BC} \times \overrightarrow{BD}\right|}$$

$$\rightarrow \cos\theta = \frac{(-4i + j - k) \cdot (-6i - 3j + 3k)}{|-4i + j - k|\,|-6i - 3j + 3k|}$$

$$\rightarrow \cos\theta = \frac{24 - 3 - 3}{\sqrt{18}\,\sqrt{54}}$$

$$\rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \text{(Ans for c)}$$

7. Matrices

7.1. Standard Operations

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$

- In general, \text{AB} \n eq \text{BA}
- For square matrices, $A \times A \times A \times A \ldots \times A = A^n$
- The identity matrix is a square matrix in the form

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Has property such that $AA^{-1} = A^{-1}A = I$

7.2. Inverse Matrices

- For 2×2 matrices if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\mathbf{A}^{-1}, =, \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- For $n \times n$ matrices we can use row operations on an augmented matrix of the form:

$$\begin{pmatrix} a & b & c & \vdots & 1 & 0 & 0 \\ d & e & f & \vdots & 0 & 1 & 0 \\ g & h & i & \vdots & 0 & 0 & 1 \end{pmatrix}$$

- For any two square matrices A and B

$$(AB)^{-1} = B^{-1}A^{-1}$$

- A matrix without an inverse is known as singular
- A matrix with an inverse is non-singular

7.3. Determinants

- The determinant of 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is calculated as:

$$\det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- When the determinant of a matrix is 0, the matrix will be singular.
- The value of the determinant changes by factor k when row operations of $r_i \rightarrow kr_i + mr_j$ are used.
- The value of the determinant is also the factor increase of the area, or volume, when the matrix is used as a transformation.
- For two matrices A and B

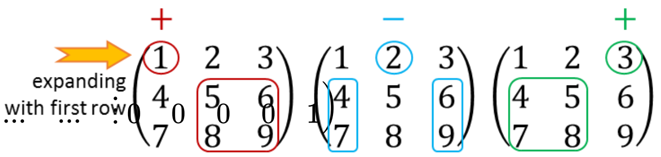
$$\det(AB) = \det(BA) = \det(A) \times \det(B)$$

Example: Find the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Solution:

Laplace expansion using the first row and applying the signs:



$$= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

Find the determinant of each matrix

$$[1 \times (45 - 48)] - [2 \times (36 - 42)] + [3 \times (32 - 35)]$$

Hence find the determinant of the 3 by 3 matrix

$$(-3) - (-12) + (-9) = 0$$

7.4. Transformations

- The following transformations are for 2×2 matrices.

Transformation	Matrix
Stretch by a scale factor of factor **k** in the x-direction	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Stretch by a scale factor of factor **k** in the y-direction	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement with center of enlargement the origin by a scale factor of factor **k**	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Reflection in the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Rotation about the origin by θ in the anticlockwise direction	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

7.5. Invariant Lines

- For 2-dimensional cases, use $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t \\ \text{mt} \end{pmatrix} = \begin{pmatrix} T \\ \text{mT} \end{pmatrix}$ to determine two equations of the form $at + bmt = T$ and $ct + dmt = mT$. Divide to get $\frac{a+bm}{c+dm} = \frac{1}{m}$, then solve for value(s) of m to find invariant line(s) of the transformation in the form $y = mx$.

{SP20-P03} Question 5:
The matrix A is given by

$$\begin{pmatrix} 5 & k \\ -3 & -4 \end{pmatrix}$$

a) Find the value of k for which A is singular it is now given $k = 6$ so that $A = \begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}$
b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by A
c) The triangle DEF in the x-y plane is transformed by A onto triangle PQR.
i) Given that the area of triangle DEF is $10, cm^2$, find the area of triangle PQR.
ii) Find the matrix which transforms triangle PQR onto triangle DEF.

Solution:

When $A$is singular, means that its determinant is equal to 0. Using the equation $ad - bc$, we find its det.

$$5(-4) - k(-3) = 0$$

$$k = \frac{20}{3} \quad (\text{Ans for a})$$

Use the equation to find invariant lines shown is section 7.5:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} t \\ \text{mt} \end{pmatrix} = \begin{pmatrix} T \\ \text{mT} \end{pmatrix}$$

Substitute values in:

$$\begin{pmatrix} 5 & 6 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} t \\ \text{mt} \end{pmatrix} = \begin{pmatrix} T \\ \text{mT} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 5t + 6\text{mt} \\ -3t - 4\text{mt} \end{pmatrix} = \begin{pmatrix} T \\ \text{mT} \end{pmatrix}$$

We now have two equations:

$$5t + 6mt = T \qquad -3t - 4mt = \text{mT}$$

Divide the 1st equation with the 2nd equation:

$$\frac{5 + 6m}{-3 - 4m} = \frac{1}{m}$$

Solve for m:

$$6m^2 + 9m + 3 = 0$$

$$m*1 = -1 \qquad m*2 = \frac{1}{2}$$

Therefore, the answers are:

$$y = -x \qquad y = \frac{1}{2}x \quad (\text{Ans for b})$$

Since determinants shows the factor increase of area/volume, we will find the determinant of A.

$$5(-4) - 6(-3) = -2$$

Therefore, ignoring the sign, the area of the new triangle is:

$$2 \times 10 = 20cm^2 \quad (\text{Ans for ci})$$

Finding the matrix that transforms the new triangle back to the old triangle just means that we must find the inverse of the matrix:

$$A^{-1}, =, \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1}, = \frac{1}{-2}\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2 & 3 \\ -\frac{3}{2} & -\frac{5}{2} \end{pmatrix} \quad (\text{Ans for c ii})$$

CAIE AS LEVEL
Further Maths (9231)

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